An upper bound for the number of arithmetical structures on a graph

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Overview

Today I will

- Define and give examples of arithmetical structures
- Describe a construction which associates an arithmetical structure on a graph to another one on a *smaller* graph
- Present a new result (joint with Tomer Reiter)

Theorem (K., Reiter [KR20])

Let G be a connected, undirected graph on n vertices, with no loops but possible multiedges. Then the following is an upper bound for the number of arithmetical structures on G.

$$\#A(G) \leq \frac{n!}{2} \cdot \#E(G)^{2^{n-2}-1} \cdot \#E(G)^{2^{n-1} \cdot \frac{1.538 \log(2)}{(n-1)\log(2)+\log(\log(\#E(G)))}}.$$

- Refine for complete graphs and make connections with the Egyptian fraction problem (time permitting)

Setup

Let G be a connected, undirected graph with n vertices and edge set $E(G)$ (possibly multiedges!).

Assume the vertices come ordered $v_1, \ldots, v_n$.

Let δ_{ij} denote the number of edges between v_i and v_j. The **adjacency matrix** of G is $A = (\delta_{ij})$.

Assume that G has no loops: $\delta_{ii} = 0$ for all i.

Note: the assumptions that G is connected and has no loops aren't that bad to fix.

Arithmetical structures

An **arithmetical structure on** G is a pair of n-tuples $(\boldsymbol{r}, \boldsymbol{d})$ of natural numbers satisfying

$$\begin{aligned} r_1 d_1 &= r_2\delta_{12} + \cdots + r_n\delta_{1n} \\ &\vdots \\ r_n d_n &= r_1\delta_{1n} + \cdots + r_{n-1}\delta_{(n-1)n} \end{aligned}$$

and

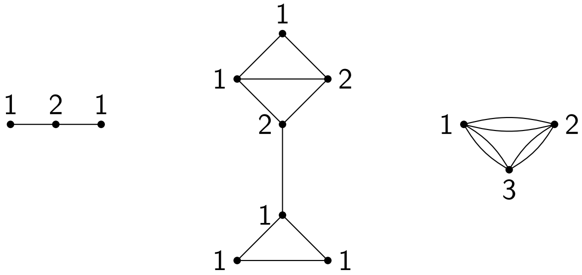
$$\gcd(r_1, \ldots, r_n) = 1.$$

It is often convenient to write $D = \operatorname{diag}(\boldsymbol{d})$, in which case we have $(D - A)\boldsymbol{r} = \boldsymbol{0}$.

Some examples

We can always set $r_i = 1$ and $d_i = \deg v_i$ for all i. In this case, the matrix $D - A$ is known as the **Laplacian**.

Some more interesting examples (with only the $\boldsymbol{r}$ values labeled):



Counting arithmetical structures

Let $A(G)$ denote the set of arithmetical structures on G.

Question

For a given graph G, how large is $\#A(G)$?

What do we know?

- Finiteness: $\#A(G) < \infty$, [Lor89]
- Paths: $\#A(P_n) = C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$, [BCC+18]
- Cycles: $\#A(C_n) = (2n-1)C_{n-1} = \binom{2n-1}{n-1}$, [BCC+18]
- Bidents, doubled edges: bounds and/or asymptotics, [GW19], [ABDL+20]

Counting arithmetical structures

The graphs studied so far have lots of regularity.

What about a general graph? Can we count (or bound) $\#A(G)$ in terms of **only** n and $\#E(G)$?

Theorem (K., Reiter [KR20])

$$\#A(G) \leq \frac{n!}{2} \cdot \#E(G)^{2^{n-2}-1} \cdot \#E(G)^{2^{n-1} \cdot \frac{1.538 \log(2)}{(n-1)\log(2)+\log(\log(\#E(G)))}}.$$

Proof idea

Given an arithmetical structure $(\boldsymbol{r}, \boldsymbol{d})$ on G with n vertices, cook up $(\boldsymbol{r}', \boldsymbol{d}')$ on G' with $n-1$ vertices.

Use induction to reduce to the case of a graph with two vertices.

Graphs with two vertices

Let G be a graph with $n = 2$ vertices and m edges.

Let $\sigma_0(n) = \#\{\text{positive divisors of } n\}$ denote the divisor function.

Lemma

$\#A(G) = \sigma_0(m^2)$.

Proof.

An arith. struct. on G is a *coprime* pair (r_1, r_2) such that $r_1 \mid mr_2$ and symmetrically $r_2 \mid mr_1$ (here the d_i are implicit). The map

$$(r_1, r_2) \mapsto \frac{mr_2}{r_1}$$

is a bijection from such pairs to divisors of m^2. □

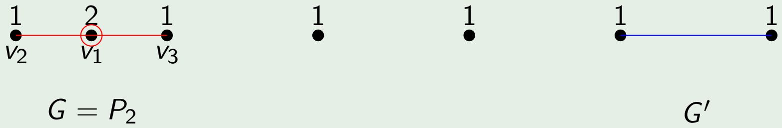
The construction by example

Start with an arithmetical structure $(\boldsymbol{r}, \boldsymbol{d})$ on G. We will remove vertex v_1 and construct a graph G' on the remaining vertices:

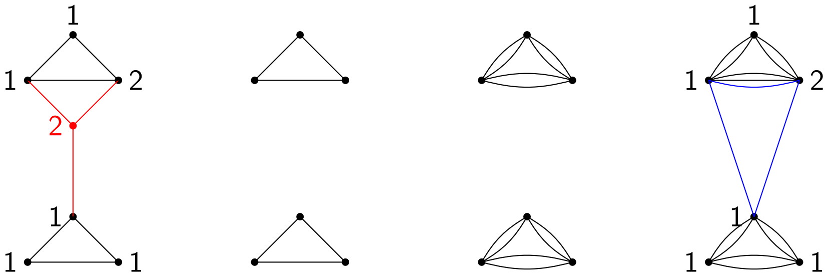
1. Remove v_1 and all incident edges.
2. Replace remaining edges by d_1 copies.
3. For remaining distinct v_i, v_j, add $\delta_{1i}\delta_{1j}$ edges.
4. Obtain $\boldsymbol{r}'$ by deleting r_1 and scaling if necessary

Example

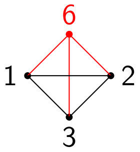
Vertices are labeled with r_i values.

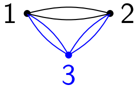


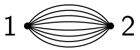
The construction by example



The construction by example







The construction by example

This construction is a generalization of the **smoothing** process in [BCC+18] and [GW19] for paths (with and without a doubled edge) and cycles.

When $d_1 = 1$, the construction is inverse to the **blow up** construction in [Lor89] and generalizes previous observations by [CV18] about the **clique-star transform**.

In particular, when $d_1 = 1$, the critical group is unchanged,

$$K(G, \mathbf{r}) \simeq K(G', \mathbf{r}').$$

Unanswered question

Can we say anything more generally about how this transformation affects the critical group?

Completing proof of main theorem

1. Let v_i be the vertex with maximal r_i value.
2. Let $G'(i, d_i)$ denote the graph obtained by our construction for some value of d_i.
3. Apply induction on the number of vertices, taking care to keep track of how the number of edges grows.
4. For base case of $n = 2$, use a *monotonically increasing* upper bound for σ_0, e.g. $\sigma_0(m) \leq m^{\frac{1.538\log(2)}{\log\log(m)}}$ [Nic88].

Theorem (K., Reiter [KR20])

$$\#A(G) \leq \frac{n!}{2} \cdot \#E(G)^{2^{n-2}-1} \cdot \#E(G)^{2^{n-1} \cdot \frac{1.538\log(2)}{(n-1)\log(2)+\log(\log(\#E(G)))}}.$$

Arithmetical structures on complete graphs

Let K_n denote the **complete graph on** n **vertices**.

Let mK_n denote the graph on n vertices with m *edges between each vertex pair.*

Let $A_{\text{dec}}(mK_n)$ denote the subset of $A(mK_n)$ with *decreasing* r-values, $r_1 \geq r_2 \geq \cdots \geq r_n$.

Our construction associates an arith. struct. on $(m^2 + d_1 m)K_{n-1}$.

We can use this to compute all the arithmetical structures on mK_n when m and n are small.

A comparison

n	m	$\#A_{\text{dec}}(mK_n)$	Our best bound
3	1	3	20.60
3	2	10	56.46
3	3	21	127.58
3	4	28	229.66
3	5	36	362.62
3	6	57	526.39
3	7	42	720.90
3	8	70	946.06
3	9	79	1201.76
3	10	96	1487.91
3	100	1106	142796.93
3	101	164	145584.07
4	1	14	688.99
4	2	108	23028.32
4	3	339	173664.01
4	4	694	717812.26
4	5	1104	2141953.95
4	6	1816	5209709.25
4	7	2021	11012969.52
4	8	3363	21019441.99
4	9	4053	37117341.07
4	10	5370	61657730.38
5	1	147	8567815.81

Connections to Egyptian fractions

Theorem

$A(mK_n)$ is in bijection with primitive $(x_1, \ldots, x_n)$ satisfying

$$\frac{1}{m} = \frac{1}{x_1} + \cdots + \frac{1}{x_n}.$$

Solutions to such equations are known as **Egyptian fractions**.

Corollary (Browning–Elsholtz [BE11], Elsholtz–Planitzer [EP20])

Let $n \geq 3$, $m \geq 1$, and fix $\epsilon \geq 0$. Then

$$\#A_{dec}(mK_3) \ll_\epsilon m^{\frac{3}{5}+\epsilon}$$

$$\#A_{dec}(mK_4) \ll_\epsilon m^{\frac{28}{17}+\epsilon}$$

$$\#A_{dec}(mK_n) \ll_\epsilon (nm)^\epsilon \left(n^{4/3} m^2\right)^{\frac{28}{17} 2^{n-5}}$$

An asymptotic improvement but lacking explicit constants!

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